

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

Level - 1	Daily Tutorial Sheet - 5
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$$\begin{aligned}
 61.(C) \quad \sum_{r=1}^{18} \cos^2(5r)^\circ &= \cos^2 5^\circ + \cos^2 10^\circ + \dots + \cos^2 85^\circ + \cos^2 90^\circ \\
 &= (\cos^2 5^\circ + \cos^2 85^\circ) + (\cos^2 10^\circ + \cos^2 80^\circ) + \dots + (\cos^2 40^\circ + \cos^2 50^\circ) + \cos^2 45^\circ \\
 &= (\cos^2 5^\circ + \sin^2 5^\circ) + (\cos^2 10^\circ + \sin^2 10^\circ) + \dots + (\cos^2 40^\circ + \sin^2 40^\circ) + \cos^2 45^\circ \\
 &= \underbrace{1 + 1 + 1 + \dots + 1}_{8 \text{ times}} + \frac{1}{2} = 8 + \frac{1}{2} = \frac{17}{2}
 \end{aligned}$$

$$62.(A) \quad a \sin x + b \{\cos(x + \theta) + \cos(x - \theta)\} = d \Rightarrow a \sin x + 2b \cos x \cos \theta = d \quad \dots(i)$$

$$\text{Let } f(x) = a \sin x + 2b \cos x \cos \theta$$

$$\text{Now, } -\sqrt{a^2 + 4b^2 \cos^2 \theta} \leq a \sin x + 2b \cos x \cos \theta \leq \sqrt{a^2 + 4b^2 \cos^2 \theta}$$

$$\Rightarrow |a \sin x + 2b \cos x \cos \theta| \leq \sqrt{a^2 + 4b^2 \cos^2 \theta}$$

$$\text{From (i) } |d| \leq \sqrt{a^2 + 4b^2 \cos^2 \theta} \Rightarrow \cos^2 \theta \geq \frac{d^2 - a^2}{4b^2} \Rightarrow |\cos \theta| \geq \frac{\sqrt{d^2 - a^2}}{2|b|}$$

$$\begin{aligned}
 63.(A) \quad \sqrt{1 + \cos x} &= \sqrt{2 \cos^2 \frac{x}{2}} = \sqrt{2} \left| \cos \frac{x}{2} \right| \text{ and } \sqrt{1 - \cos x} = \sqrt{2 \sin^2 \frac{x}{2}} = \sqrt{2} \left| \sin \frac{x}{2} \right| \\
 \Rightarrow \frac{\sqrt{1 + \cos x} + \sqrt{1 - \cos x}}{\sqrt{1 + \cos x} - \sqrt{1 - \cos x}} &= \frac{\left| \cos \frac{x}{2} \right| + \left| \sin \frac{x}{2} \right|}{\left| \cos \frac{x}{2} \right| - \left| \sin \frac{x}{2} \right|} = \frac{-\cos \frac{x}{2} + \sin \frac{x}{2}}{-\cos \frac{x}{2} - \sin \frac{x}{2}} = \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} = \frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}} \\
 &= \tan \left(\frac{\pi}{4} - \frac{x}{2} \right) = \cot \left(\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{x}{2} \right) \right) = \cot \left(\frac{\pi}{4} + \frac{x}{2} \right)
 \end{aligned}$$

$$64.(D) \quad \text{Consider: } \frac{\cos x}{a} = \frac{\sin x}{b} \Rightarrow \tan x = \frac{b}{a}$$

$$\text{Now, } \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1 - \frac{b^2}{a^2}}{1 + \frac{b^2}{a^2}} = \frac{a^2 - b^2}{a^2 + b^2} \Rightarrow a \cos 2x = \frac{a(a^2 - b^2)}{a^2 + b^2}$$

$$\text{Again, } \sin 2x = \frac{2 \tan x}{1 + \tan^2 x} = \frac{2 \frac{b}{a}}{1 + \frac{b^2}{a^2}} = \frac{2ab}{a^2 + b^2}$$

$$\Rightarrow b \sin 2x = \frac{2ab^2}{a^2 + b^2} \Rightarrow a \cos 2x + b \sin 2x = \frac{a^3 + ab^2}{a^2 + b^2} = a$$

$$65.(B) \quad \cot^2 x = \cot(x - y) \cdot \cot(x - z)$$

$$\Rightarrow \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x} \right) \left(\frac{\cot x \cot z + 1}{\cot z - \cot x} \right)$$

$$\Rightarrow \cot^3 x \cdot (\cot y + \cot z) + \cot x (\cot y + \cot z) + 1 - \cot^4 x = 0$$

$$\begin{aligned}
 &\Rightarrow \cot x (\cot y + \cot z) (1 + \cot^2 x) + (1 - \cot^2 x) (1 + \cot^2 x) = 0 \\
 &\Rightarrow (\cot x (\cot y + \cot z) + (1 - \cot^2 x)) = 0 \quad [\text{As } \cot^2 x \neq -1] \\
 &\Rightarrow \frac{\cot^2 x - 1}{2 \cot x} = \frac{1}{2} (\cot y + \cot z) = \cot 2x
 \end{aligned}$$

66.(B) $a \cos x + b \sin x = c$

$$\Rightarrow \frac{a \left(1 - \tan^2 \frac{x}{2}\right)}{1 + \tan^2 \frac{x}{2}} + \frac{2b \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = c \Rightarrow (c + a) \tan^2 \frac{x}{2} - 2b \tan \frac{x}{2} + c - a = 0$$

Let $\tan \frac{x_1}{2}$ and $\tan \frac{x_2}{2}$ are roots of above equation.

$$\Rightarrow \tan \frac{x_1}{2} + \tan \frac{x_2}{2} = \frac{2b}{c+a}, \tan \frac{x_1}{2} \cdot \tan \frac{x_2}{2} = \frac{c-a}{c+a} \Rightarrow \tan \left(\frac{x_1 + x_2}{2} \right) = \frac{\frac{2b}{c+a}}{1 - \frac{c-a}{c+a}} = \frac{2b}{2a} = \frac{b}{a}$$

67.(B) Since the value of $\sec^2 \theta$ is always ≥ 1

$$\therefore \frac{4xy}{(x+y)^2} \geq 1 \Rightarrow (x+y)^2 - 4xy \leq 0 \Rightarrow (x-y)^2 \leq 0 \Rightarrow (x-y)^2 = 0 \text{ as } (x-y)^2 \text{ can never be negative}$$

$$\Rightarrow x = y$$

Here, the fraction $\frac{4xy}{(x+y)^2}$ is not defined for $x+y=0$

$$\Rightarrow x+y \neq 0 \Rightarrow x+x \neq 0 \Rightarrow 2x \neq 0 \text{ or } x \neq 0 \quad \text{Hence } x=y, x \neq 0.$$

68.(D) From the given relationship we have $2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} = a$ and $2 \sin \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} = b$

$$\text{On dividing we get: } \tan \frac{\alpha+\beta}{2} = \frac{b}{a} \Rightarrow \tan \theta = \frac{b}{a} \quad \left[\because \theta = \frac{\alpha+\beta}{2} \right]$$

$$\text{So that } \cos 2\theta = \frac{1-b^2/a^2}{1+b^2/a^2} = \frac{a^2-b^2}{a^2+b^2} \text{ and } \sin 2\theta = \frac{2ab}{a^2+b^2} \quad \therefore \sin 2\theta + \cos 2\theta = \frac{a^2-b^2+2ab}{a^2+b^2}$$

69.(B) We have, $\cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta$

$$\Rightarrow (-2b)^2 - 4b = (4)^2 - 4 \times 2 \Rightarrow b^2 - b - 2 = 0 \Rightarrow b = -1, 2$$

70.(A) Statement 2 is a standard trigonometric identity.

Statement 1

$$\sec \theta - \tan \theta = \frac{\sec^2 \theta - \tan^2 \theta}{\sec \theta + \tan \theta} = \frac{1}{\sec \theta + \tan \theta} \quad [\text{Using the result from Statement 2}]$$

$$\text{i.e., } \sec \theta - \tan \theta = \frac{1}{m} \quad \dots\dots\dots(\text{i})$$

$$\text{Also } \sec \theta + \tan \theta = m \quad \dots\dots\dots(\text{ii})$$

$$\text{Adding (i) and (ii), we get: } 2 \sec \theta = m + \frac{1}{m} = \frac{m^2 + 1}{m} \quad \dots\dots\dots(\text{iii})$$

Subtracting (i) and (ii), we get: $2 \tan \theta = m - \frac{1}{m} = \frac{m^2 - 1}{m}$ (iv)

On dividing (iii) from (iv), we obtain $\sin \theta = \frac{m^2 - 1}{m^2 + 1}$

71.(A) Statement 2: If t is a +ve number, then: $t + \frac{1}{t} = \left(\sqrt{t} - \frac{1}{\sqrt{t}} \right)^2 + 2 \geq 2$

Now, $x^2 + \frac{1}{x^2} \geq 2$, $x \neq 0$ and $\cos \theta \in [-1, 1]$. Therefore $x^2 + \frac{1}{x^2}$ cannot be equal to $\cos \theta$ for any θ .

72.(B) Statement 1 is true

Statement 2

$\sin x$ increases in $\left[0, \frac{\pi}{2} \right]$

Also, $\sin x = \sin(\pi - x) \forall x \in R$

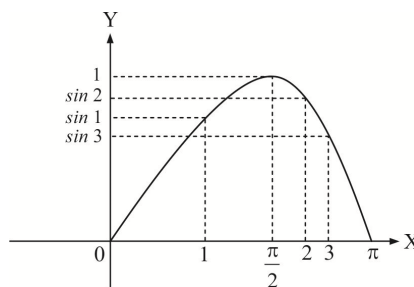
$$1 \in \left[0, \frac{\pi}{2} \right]$$

$$2, 3 \in \left(\frac{\pi}{2}, \pi \right)$$

$$\Rightarrow \pi - 2, \pi - 3 \in \left(0, \frac{\pi}{2} \right)$$

Also, $\pi - 3 < 1 < \pi - 2$

$$\Rightarrow \sin(\pi - 3) < \sin 1 < \sin(\pi - 2) \Rightarrow \sin 3 < \sin 1 < \sin 2$$



73.(D) Statement 2 is a standard result.

However, the statement 1 is not true as the equation $a \sin \theta + b \cos \theta = c$ has a possible real solution in only if $c^2 \leq a^2 + b^2$

74.(AD) $\sec \theta + \tan \theta = 1$

$$\text{and } \sec^2 \theta - \tan^2 \theta = 1 \Rightarrow (\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = 1$$

$$\Rightarrow \sec \theta - \tan \theta = 1 \Rightarrow \sec \theta = 1 \text{ and } \tan \theta = 0$$

$$\text{Let } f(x) = (a - 2b + c)x^2 + (b - 2c + a)x + (c - a + 2b)$$

$$\text{Since } f(1) = 0$$

Roots of the given equation are 1 and $\frac{c - 2a + b}{a - 2b + c}$

$$\Rightarrow \sec \theta \text{ and } \cos \theta \text{ are its roots } (\because \text{both are } 1).$$

75.(D) $\tan x + \cot x = 2 \Rightarrow \tan^2 x - 2 \tan x + 1 = 0 \Rightarrow (\tan x - 1)^2 = 0 \Rightarrow \tan x = 1 \Rightarrow \sin^2 x = \cos^2 x = \frac{1}{2}$

$$\Rightarrow \sin^{2n} x + \cos^{2n} x = \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$$